An Algebraic Formulation of Level One Wess-Zumino-Witten Models

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Abstract

The highest weight modules of the chiral algebra of orthogonal WZW models at level one possess a realization in fermionic representation spaces; the Kac-Moody and Virasoro generators are represented as unbounded limits of even CAR algebras. It is shown that the representation theory of the underlying even CAR algebras reproduces precisely the sectors of the chiral algebra. This fact allows to develop a theory of local von Neumann algebras on the punctured circle, fitting nicely in the Doplicher-Haag-Roberts framework. The relevant localized endomorphisms which generate the charged sectors are explicitly constructed by means of Bogoliubov transformations. Using CAR theory, the fusion rules in terms of sector equivalence classes are proven.

1 Introduction

In local quantum field theory one considers a Hilbert space $\mathcal{H}$ of physical states which decomposes into orthogonal subspaces $\mathcal{H}_J$ (superselection sectors) so that observables do not make transitions between the sectors. The subspaces $\mathcal{H}_J$ carry inequivalent, irreducible representations of the observable algebra $\mathfrak{A}_{\text{loc}}$, possibly with some multiplicities (see [15] for an overview). Among the superselection sectors, there is a distinguished sector $\mathcal{H}_0$ which contains the vacuum vector $|\Omega_0\rangle$ and carries the vacuum representation π_0.

The starting point in the algebraic approach to quantum field theory [8, 9, 15] is the quasilocal observable algebra $\mathfrak{A}_{\text{loc}}$ which is usually defined as the C^*-inductive limit of the net of local von Neumann algebras $\{\mathcal{R}(\mathcal{O}), \mathcal{O} \in \mathfrak{K}\}$, where $\mathfrak{K}$ denotes the set of open double cones in D dimensional Minkowski space. The Doplicher-Haag-Roberts (DHR) criterion selects only those representations π_J which become equivalent to the vacuum representation in restriction to the algebra $\mathfrak{A}(\mathcal{O}')$ of the causal complement $\mathcal{O}'$ for some sufficiently large double cone $\mathcal{O}$. ($\mathfrak{A}(\mathcal{O}')$ is the norm-closure of the union of all $\mathcal{R}(\mathcal{O}_1)$, $\mathcal{O}_1 \subset \mathcal{O}'$.) The DHR selection criterion leads to the description of sectors (unitary equivalence classes $[\pi_J]$) by localized endomorphisms ϱ_J of $\mathfrak{A}_{\text{loc}}$, $\varrho_J(A) = A$ for all $A \in \mathfrak{A}(\mathcal{O}')$, so that all physical information is contained in the vacuum sector: $\pi_J \simeq \pi_0 \circ \varrho_J$. This leads to the important fact that DHR sectors can be composed,

i.e. one has a product of sectors $[\pi_J] \times [\pi_{J'}] = [\pi_0 \circ \varrho_J \varrho_{J'}]$; so one can derive fusion rules, given by the composition of localized endomorphisms.

While being an abstract and mathematically rigorous setting, the DHR analysis appeared to be difficult to be applied to concrete quantum field theory models. In the last years two-dimensional conformal quantum field theory (CQFT) turned out to be a hopeful area of application. Many of the features of CQFT appear to be closely related to the abstract structures of the algebraic setting [11]. For instance, the chiral algebra can be considered as observable algebra and its highest weight modules play the role of superselection sectors. Then the conformal fusion rules seem to be the natural counterpart of the product of sectors realized by the composition of localized endomorphisms in the algebraic approach. However, there is no mathematically precise prescription known how to translate the objects of CQFT into the framework of algebraic quantum field theory. Indeed, the attempts to incorporate CQFT models in this framework sometimes seemed to require some deviations of the DHR program; operator algebras of observables and endomorphisms were constructed in [17, 12], however, the use of non-localized endomorphisms, non-faithful representations and algebras containing unbounded elements violated the canonical DHR framework. As a consequence, the composition of non-localized endomorphisms could not be generalized to the fusion of sectors i.e. of equivalence classes $[\pi_J]$ of representations.

Inspired by a paper of Fuchs, Ganchev and Vecsernyés [12], we give a formulation close to the DHR program for a special class of conformal models, the $\mathbf{so}(N)$-Wess-Zumino-Witten (WZW) models at level one. Because the Kac-Moody and Virasoro generators of their chiral algebra can be built as infinite series of fermion bilinears in representation spaces of the canonical anticommutation relations (CAR) one expects that the sectors of the chiral algebra (highest weight modules) find their analogue in irreducible representations of the underlying even CAR algebras. Using results of Araki's selfdual CAR algebra, we show that there is indeed a one-to-one correspondence. So we can restrict our attention to algebras of bounded operators and get a theory of local C^*-algebras $\mathcal{A}(I)$ on the circle ($I \subset S^1$). We then construct localized endomorphisms by means of Bogoliubov transformations and show that we can extend our representations and endomorphisms to a net of local von Neumann algebras $\mathcal{R}(I)$ which generate a quasilocal algebra $\mathfrak{A}_{\rm loc}$ on the punctured circle. Thus, up to the replacement of double cones by intervals, we do not leave the DHR framework. Hence now one can deduce fusion rules in terms of sector equivalence classes by computing the composition of special representative localized endomorphisms. Using our examples, we indeed rediscover the well-known WZW fusion rules.

2 The Chiral Algebra of Level 1 WZW Models

As already mentioned, in two-dimensional CQFT the analog role of the observable algebra is played by the chiral algebra which is the chiral half of the symmetry algebra. In WZW theory we consider here, the chiral algebra $\mathcal{L}$ is the semi-direct product of an untwisted Kac-Moody algebra $\widehat{\mathbf{so}}(N)_k$, generated by "currents" J^α_m, $\alpha = 1, 2, \ldots, \frac{1}{2}N(N-1)$, $m \in \mathbb{Z}$, $N \in \mathbb{N}$ fixed, and the Virasoro algebra Vir_c, generated by L_n, $n \in \mathbb{Z}$; in our models the level and central charge

are fixed to be $k = 1$ and $c = N/2$,

$$\mathcal{L} = \widehat{\mathbf{so}}(N)_1 \rtimes \mathsf{Vir}_{N/2}. \tag{1}$$

Now let T^α, $\alpha = 1, 2, \ldots, \frac{1}{2}N(N-1)$ be real antisymmetric matrix generators of the finite-dimensional Lie algebra $\mathbf{so}(N)$ in the defining (vector) representation. We denote by $f_\gamma^{\alpha\beta}$ the structure constants and by $\kappa^{\alpha\beta}$ the Cartan-Killing form of $\mathbf{so}(N)$, i.e. $[T^\alpha, T^\beta] = f_\gamma^{\alpha\beta} T^\gamma$ (summation over γ) and $\kappa^{\alpha\beta} = \mathrm{tr}(T^\alpha T^\beta)$. Then the commutation relations read

$$\begin{aligned}
[J_m^\alpha, J_n^\beta] &= f_\gamma^{\alpha\beta} J_{m+n}^\gamma + m\kappa^{\alpha\beta}\delta_{m,-n}, && (2)\\
[L_m, L_n] &= (m-n)L_{m+n} + \frac{N}{24}(m^3 - m)\delta_{m,-n}, && (3)\\
[L_m, J_n^\alpha] &= -nJ_{m+n}^\alpha. && (4)
\end{aligned}$$

The representation theory of $\mathcal{L}$ is well known [13, 16], for N even, i.e. $N \in 2\mathbb{N}$, there are four different integrable highest weight modules, the basic (denoted by 0 or 1), the vector (v) and two different spinor modules (s and c); for N odd, i.e. $N \in 2\mathbb{N}_0 + 1$ there is only the basic $(0, 1)$, the vector (v) and one spinor (σ) module. (The case $N = 1$ reproduces formally the Ising model.) These modules correspond to L_0-eigenvalues $h_0 = 0$, $h_\mathrm{v} = 1/2$ and $h_\mathrm{s} = h_\mathrm{c} = h_\sigma = N/16$ of their highest weight vectors. We will realize these modules in representation spaces $\mathcal{H}_\mathrm{NS}$ and $\mathcal{H}_\mathrm{R}$ of CAR. The $\mathsf{Vir}_{N/2}$ and $\widehat{\mathbf{so}}(N)_1$ Kac-Moody generators then become expressions as infinite series of normal ordered fermion bilinears, that means they are represented as unbounded limits of even CAR algebras. For this purpose we introduce N Majorana fields ψ^i, $i = 1, 2, \ldots, N$, on the circle S^1, with hermiticity condition

$$(\psi^i(z))^* = \psi^i(z)$$

and satisfying anticommutation relations

$$\{\psi^i(z), \psi^j(z')\} = 2\pi\mathrm{i}z\, \delta^{i,j}\delta(z - z').$$

Consider an N-component L^2-function on the circle, $f = (f^i)_{i=1,2,\ldots,N} \in \mathcal{K} = L^2(S^1; \mathbb{C}^N)$. Define an antiunitary involution Γ by component-wise complex conjugation, $\Gamma f = (\overline{f^i})_{i=1,2,\ldots,N}$. Then smeared objects

$$B(f) = \sum_{j=1}^{N} \oint_{S^1} \frac{\mathrm{d}z}{2\pi\mathrm{i}z} f^j(z)\psi^j(z)$$

obey the defining relations of the canonical generators of Araki's [1, 2] selfdual CAR algebra $\mathcal{C}(\mathcal{K}, \Gamma)$,

$$\{B(f)^*, B(g)\} = \langle f, g\rangle \mathbf{1}, \qquad B(f)^* = B(\Gamma f), \qquad f, g \in \mathcal{K},$$

with the canonical scalar product $\langle\cdot,\cdot\rangle$ on $\mathcal{K}$,

$$\langle f, g\rangle = \sum_{j=1}^{N} \oint_{S^1} \frac{\mathrm{d}z}{2\pi\mathrm{i}z} \overline{f^j(z)} g^j(z).$$

The selfdual CAR algebra is discussed in the following section.

3 The Selfdual CAR Algebra

Let $\mathcal{K}$ be some Hilbert space with an antiunitary involution Γ (complex conjugation), $\Gamma^2 = \mathbf{1}$, which obeys

$$\langle \Gamma f, \Gamma g \rangle = \langle g, f \rangle, \qquad f, g \in \mathcal{K}.$$

The selfdual CAR algebra $\mathcal{C}(\mathcal{K}, \Gamma)$ is defined to be the C^*-norm closure of the algebra which is generated by the range of a linear mapping $B : f \mapsto B(f)$, such that

$$\{B(f)^*, B(g)\} = \langle f, g \rangle \mathbf{1}, \qquad B(f)^* = B(\Gamma f), \qquad f, g \in \mathcal{K}.$$

holds. The C^*-norm satisfies

$$\|B(f)\| \leq \|f\|, \qquad f \in \mathcal{K}. \tag{5}$$

The states of $\mathcal{C}(\mathcal{K}, \Gamma)$ we are interested in are called quasifree states. By definition, a quasifree state ω fulfills for $n \in \mathbb{N}$

$$\begin{aligned} \omega(B(f_1) \cdots B(f_{2n+1})) &= 0, \\ \omega(B(f_1) \cdots B(f_{2n})) &= (-1)^{\frac{n(n-1)}{2}} \sum_{\sigma} \mathrm{sign}\sigma \prod_{j=1}^{n} \omega(B(f_{\sigma(j)}) B(f_{\sigma(n+j)})) \end{aligned}$$

where the sum runs over all permutations $\sigma \in \mathcal{S}_{2n}$ with the property

$$\sigma(1) < \sigma(2) < \cdots < \sigma(n), \qquad \sigma(j) < \sigma(j+n), \qquad j = 1, \ldots, n.$$

Clearly, quasifree states are completely characterized by their two point functions. Moreover, there is a one-to-one correspondence between the set of quasifree states and the set

$$\mathcal{Q}(\mathcal{K}, \Gamma) = \{S \in \mathfrak{B}(\mathcal{K}) \,|\, S = S^*,\ 0 \leq S \leq \mathbf{1},\ S + \overline{S} = \mathbf{1}\},$$

(we have used the notation $\overline{A} = \Gamma A \Gamma$ for bounded operators $A \in \mathfrak{B}(\mathcal{K})$) given by the formula

$$\omega(B(f)^* B(g)) = \langle f, S g \rangle. \tag{6}$$

So it is convenient to denote the quasifree state characterized by Eq. (6) by ω_S. The projections in $\mathcal{Q}(\mathcal{K}, \Gamma)$ are called basis projections or polarizations. For a basis projection P, the state ω_P is pure and is called a Fock state. The corresponding GNS representation $(\mathcal{H}_P, \pi_P, |\Omega_P\rangle)$ is irreducible, it is called the Fock representation. The space $\mathcal{H}_P$ can be canonically identified with the antisymmetric Fock space $\mathcal{F}_-(P\mathcal{K})$. There is an important quasiequivalence criterion for GNS representations of quasifree states. Quasiequivalence will be denoted by "$\approx$" and unitary equivalence by "$\simeq$". Let us denote by $\mathfrak{J}_2(\mathcal{K})$ the ideal of Hilbert-Schmidt operators in $\mathfrak{B}(\mathcal{K})$ and for $A \in \mathfrak{B}(\mathcal{K})$ by $[A]_2$ its Hilbert-Schmidt equivalence class $[A]_2 = A + \mathfrak{J}_2(\mathcal{K})$. Araki proved [1, 2]

Theorem 3.1 *For quasifree states ω_{S_1} and ω_{S_2} of $\mathcal{C}(\mathcal{K}, \Gamma)$ we have quasiequivalence $\pi_{S_1} \approx \pi_{S_2}$ if and only if $[S_1^{\frac{1}{2}}]_2 = [S_2^{\frac{1}{2}}]_2$.*

Next we define the set

$$\mathcal{I}(\mathcal{K},\Gamma) = \{V \in \mathfrak{B}(\mathcal{K}) \,|\, V^*V = \mathbf{1},\ V = \overline{V}\}$$

of Bogoliubov operators. Bogoliubov operators $V \in \mathcal{I}(\mathcal{K},\Gamma)$ induce unital $*$-endomorphisms ϱ_V of $\mathcal{C}(\mathcal{K},\Gamma)$, defined by their action on the canonical generators,

$$\varrho_V(B(f)) = B(Vf).$$

Moreover, if $V \in \mathcal{I}(\mathcal{K},\Gamma)$ is surjective (i.e. unitary), then ϱ_V is an automorphism. A quasifree state, composed with a Bogoliubov endomorphism is again a quasifree state, namely we have $\omega_S \circ \varrho_V = \omega_{V^*SV}$. In the following we are interested in representations of the form $\pi_P \circ \varrho_V$ instead of GNS representations π_{V^*PV} of states $\omega_{V^*PV} = \omega_P \circ \varrho_V$. Indeed, the former are multiples of the latter, in particular, we have [4, 19]

$$\pi_P \circ \varrho_V \simeq 2^{N_V} \pi_{V^*PV}, \qquad N_V = \dim(\ker V^* \cap P\mathcal{K}). \tag{7}$$

Thus, the identification of the Hilbert-Schmidt equivalence class $[(V^*PV)^{\frac{1}{2}}]_2$ is the identification of the quasiequivalence class of $\pi_P \circ \varrho_V$. For the identification of the unitary equivalence class, we need a decomposition of $\pi_P \circ \varrho_V$ into irreducible subrepresentations which will now be elaborated. A projection $E \in \mathfrak{B}(\mathcal{K})$ with the property that $E\overline{E} = 0$ and that $\ker(E + \overline{E}) = \mathbb{C}e_0$ with a Γ-invariant unit vector $e_0 \in \mathcal{K}$ is called a partial basis projection with Γ-codimension 1. Note that E defines a Fock representation $(\mathcal{H}_E, \pi_E, |\Omega_E\rangle)$ of $\mathcal{C}((E+\overline{E})\mathcal{K},\Gamma)$. Following Araki, pseudo Fock representations $\pi_{E,+}$ and $\pi_{E,-}$ of $\mathcal{C}(\mathcal{K},\Gamma)$ are defined in $\mathcal{H}_E$ by

$$\pi_{E,\pm}(B(f)) = \pm\frac{1}{\sqrt{2}}\langle e_0, f\rangle Q_E(-1) + \pi_E(B((E+\overline{E})f)), \qquad f \in \mathcal{K}, \tag{8}$$

where $Q_E(-1) \in \mathfrak{B}(\mathcal{K})$ is the unitary, self-adjoint implementer of the automorphism α_{-1} of $\mathcal{C}(\mathcal{K},\Gamma)$ defined by $\alpha_{-1}(B(f)) = -B(f)$ (which restricts also to an automorphism of $\mathcal{C}((E+\overline{E})\mathcal{K},\Gamma)$). Pseudo Fock representations $\pi_{E,+}$ and $\pi_{E,-}$ are inequivalent and irreducible. Araki proved [1]

Lemma 3.2 *Let E be a partial basis projection with Γ-codimension 1, and let $e_0 \in \mathcal{K}$ be a Γ-invariant unit vector of $\ker(E+\overline{E})$. Define $S \in \mathcal{Q}(\mathcal{K},\Gamma)$ by*

$$S = \frac{1}{2}|e_0\rangle\langle e_0| + E. \tag{9}$$

Then a GNS representation $(\mathcal{H}_S, \pi_S, |\Omega_S\rangle)$ of the quasifree state ω_S is given by the direct sum of two inequivalent, irreducible pseudo Fock representations,

$$(\mathcal{H}_S, \pi_S, |\Omega_S\rangle) = \left(\mathcal{H}_E \oplus \mathcal{H}_E, \pi_{E,+} \oplus \pi_{E,-}, \frac{1}{\sqrt{2}}(|\Omega_E\rangle \oplus |\Omega_E\rangle)\right). \tag{10}$$

It was the observation in [6] (see also [4]) that only Fock and pseudo Fock representations appear in the decomposition of representations $\pi_P \circ \varrho_V$ if the Bogoliubov operator has finite corank.

Theorem 3.3 *Let P be a basis projection and let V be a Bogoliubov operator with $M_V = \dim \ker V^* < \infty$. If M_V is an even integer we have (with notations as above)*

$$\pi_P \circ \varrho_V \simeq 2^{\frac{M_V}{2}} \pi_{P'} \tag{11}$$

where $\pi_{P'}$ is an (irreducible) Fock representation. If M_V is odd then we have

$$\pi_P \circ \varrho_V \simeq 2^{\frac{M_V - 1}{2}} (\pi_{E,+} \oplus \pi_{E,-}) \tag{12}$$

where $\pi_{E,+}$ and $\pi_{E,-}$ are inequivalent (irreducible) pseudo Fock representations.

We define the even algebra $\mathcal{C}(\mathcal{K},\Gamma)^+$ to be the subalgebra of α_{-1}-fixpoints,

$$\mathcal{C}(\mathcal{K},\Gamma)^+ = \{x \in \mathcal{C}(\mathcal{K},\Gamma) \mid \alpha_{-1}(x) = x\}.$$

We now are interested in what happens when our representations of $\mathcal{C}(\mathcal{K},\Gamma)$ are restricted to the even algebra. For basis projections P_1, P_2, with $[P_1]_2 = [P_2]_2$, Araki and D.E. Evans [3] defined an index, taking values ± 1,

$$\operatorname{ind}(P_1, P_2) = (-1)^{\dim(P_1\mathcal{K} \cap (\mathbf{1} - P_2)\mathcal{K})}.$$

The automorphism α_{-1} leaves any quasifree state ω_S invariant. Hence α_{-1} is implemented in π_S. In particular, in a Fock representation π_P, α_{-1} extends to an automorphism $\bar{\alpha}_{-1}$ of $\pi_P(\mathcal{C}(\mathcal{K},\Gamma))'' = \mathfrak{B}(\mathcal{H}_P)$. The following proposition is taken from [2].

Proposition 3.4 *Let $U \in \mathcal{I}(\mathcal{K},\Gamma)$ be a unitary Bogoliubov operator and let P be a basis projection such that $[P]_2 = [U^*PU]_2$. Denote by $Q_P(U) \in \mathfrak{B}(\mathcal{H}_P)$ the unitary which implements ϱ_U in π_P. Then*

$$\bar{\alpha}_{-1}(Q_P(U)) = \sigma(U) Q_P(U), \qquad \sigma(U) = \pm 1. \tag{13}$$

In particular, $\sigma(U) = \operatorname{ind}(P, U^*PU)$. Moreover, given two unitaries $U_1, U_2 \in \mathcal{I}(\mathcal{K},\Gamma)$ of this type, σ is multiplicative, $\sigma(U_1U_2) = \sigma(U_1)\sigma(U_2)$.

Furthermore, one has [3, 2]

Theorem 3.5 *Restricted to the even algebra $\mathcal{C}(\mathcal{K},\Gamma)^+$, a Fock representation π_P splits into two mutually inequivalent, irreducible subrepresentations,*

$$\pi_P|_{\mathcal{C}(\mathcal{K},\Gamma)^+} = \pi_P^+ \oplus \pi_P^-. \tag{14}$$

Given two basis projections P_1, P_2, then $\pi_{P_1}^\pm \simeq \pi_{P_2}^\pm$ if and only if $[P_1]_2 = [P_2]_2$ and $\operatorname{ind}(P_1, P_2) = +1$, and $\pi_{P_1}^\pm \simeq \pi_{P_2}^\mp$ if and only if $[P_1]_2 = [P_2]_2$ and $\operatorname{ind}(P_1, P_2) = -1$.

For some real $v \in \mathcal{K}$, i.e. $\Gamma v = v$, and $\|v\| = 1$ define $U \in \mathcal{I}(\mathcal{K},\Gamma)$ by

$$U = 2|v\rangle\langle v| - \mathbf{1}. \tag{15}$$

Then ϱ_U is implemented in each Fock representation π_P by the unitary self-adjoint $Q_P(U) = \sqrt{2}\pi_P(B(v))$, since ϱ_U is implemented in $\mathcal{C}(\mathcal{K},\Gamma)$ by $q(U) =$

$\sqrt{2}B(v)$,

$$\begin{aligned} q(U)B(f)q(U) &= 2B(v)B(f)B(v) \\ &= 2\{B(v),B(f)\}B(v) - 2B(f)B(v)B(v) \\ &= 2\langle v,f\rangle B(v) - B(f) \\ &= B(2\langle v,f\rangle v - f) \\ &= B(Uf). \end{aligned}$$

Hence $\sigma(U) = -1$ and we immediately have the following

Corollary 3.6 *Let $U \in \mathcal{I}(\mathcal{K},\Gamma)$ be as in Eq. (15), then for each Fock representation π_P in restriction to $\mathcal{C}(\mathcal{K},\Gamma)^+$ we have equivalence $\pi_P^{\pm} \circ \varrho_U \simeq \pi_P^{\mp}$.*

It was proven in [6] that pseudo Fock representations $\pi_{E,+}$ and $\pi_{E,-}$ of Theorem 3.3, Eq. (12), when restricted to the even algebra, remain irreducible but become equivalent. Summarizing we obtain

Theorem 3.7 *With notations of Theorem 3.3, a representation $\pi_P \circ \varrho_V$ restricts as follows to the even algebra $\mathcal{C}(\mathcal{K},\Gamma)^+$: If M_V is even we have*

$$\pi_P \circ \varrho_V|_{\mathcal{C}(\mathcal{K},\Gamma)^+} \simeq 2^{\frac{M_V}{2}}(\pi_{P'}^+ \oplus \pi_{P'}^-) \tag{16}$$

with $\pi_{P'}^+$ and $\pi_{P'}^-$ mutually inequivalent and irreducible. If M_V is odd, then

$$\pi_P \circ \varrho_V|_{\mathcal{C}(\mathcal{K},\Gamma)^+} \simeq 2^{\frac{M_V+1}{2}}\pi \tag{17}$$

with π irreducible.

4 Construction of the Representation Spaces

Now we are ready to build our $\mathcal{L}$-modules as representation spaces of $\mathcal{C}(\mathcal{K},\Gamma)$. Recall that $\mathcal{K} = L^2(S^1;\mathbb{C}^N) \equiv L^2(S^1) \otimes \mathbb{C}^N$ in our model. So we obtain two (Fourier) orthonormal bases (ONB)

$$\left\{e_r^i, r \in \mathbb{Z} + \frac{1}{2}, i = 1,2,\ldots,N\right\}, \qquad \{e_n^i, n \in \mathbb{Z}, i = 1,2,\ldots,N\}$$

by the definition

$$e_p^i = e_p \otimes u^i, \qquad p \in \frac{1}{2}\mathbb{Z}, \qquad i = 1,2,\ldots,N,$$

where $e_p \in L^2(S^1)$ are defined by $e_p(z) = z^p$ and u^i denote the canonical unit vectors of $\mathbb{C}^N$. Consider $P_{\mathrm{NS}}, S_{\mathrm{R}} \in \mathcal{Q}(\mathcal{K},\Gamma)$, the Neveu-Schwarz operator

$$P_{\mathrm{NS}} = \sum_{i=1}^N \sum_{r \in \mathbb{N}_0 + \frac{1}{2}} |e_{-r}^i\rangle\langle e_{-r}^i|$$

is a basis projection, the Ramond operator

$$S_{\mathrm{R}} = \sum_{i=1}^N \left(\frac{1}{2}|e_0^i\rangle\langle e_0^i| + \sum_{n \in \mathbb{N}} |e_{-n}^i\rangle\langle e_{-n}^i| \right)$$

is not. Let us denote by $(\mathcal{H}_{\rm NS}, \pi_{\rm NS}, |\Omega_{\rm NS}\rangle)$ and $(\mathcal{H}_{\rm R}, \pi_{\rm R}, |\Omega_{\rm R}\rangle)$ the GNS representations of the associated quasifree states $\omega_{P_{\rm NS}}$ and $\omega_{S_{\rm R}}$, respectively, i.e. in order to avoid double indices, we write $\pi_{\rm NS}$ instead of $\pi_{P_{\rm NS}}$, $\pi_{\rm R}$ instead of $\pi_{S_{\rm R}}$ etc. Further define for $i = 1, 2, \ldots, N$ the Fourier modes

$$b_r^i = \pi_{\rm NS}(B(e_r^i)), \qquad r \in \mathbb{Z} + \frac{1}{2}; \qquad b_n^i = \pi_{\rm R}(B(e_n^i)), \qquad n \in \mathbb{Z},$$

such that we have CAR $\{b_r^i, b_s^j\} = \delta^{i,j}\delta_{r,-s}\mathbf{1}$ in $\mathcal{H}_{\rm NS}$ and $\{b_m^i, b_n^j\} = \delta^{i,j}\delta_{m,-n}\mathbf{1}$ in $\mathcal{H}_{\rm R}$. It follows that

$$b_r^i|\Omega_{\rm NS}\rangle = 0, \quad r > 0, \qquad b_n^i|\Omega_{\rm R}\rangle = 0, \quad n > 0.$$

Finite particle vectors

$$b_{-r_m}^{i_m} \cdots b_{-r_2}^{i_2} b_{-r_1}^{i_1} |\Omega_{\rm NS}\rangle, \qquad r_l \in \mathbb{N}_0 + \frac{1}{2}, \qquad i_l = 1, 2, \ldots, N, \tag{18}$$

and

$$b_{-n_m}^{i_m} \cdots b_{-n_2}^{i_2} b_{-n_1}^{i_1} |\Omega_{\rm R}\rangle, \qquad n_l \in \mathbb{N}_0, \qquad i_l = 1, 2, \ldots, N, \tag{19}$$

are total in $\mathcal{H}_{\rm NS}$ and $\mathcal{H}_{\rm R}$ i.e. finite linear combinations produce dense subspaces $\mathcal{H}_{\rm NS}^{\rm fin}$ and $\mathcal{H}_{\rm R}^{\rm fin}$, respectively. Denoting normal ordering by colons,

$$: b_p^i b_q^j : \; = \left\{ \begin{array}{cc} b_p^i b_q^j & p < 0 \\ -b_q^j b_p^i & p \geq 0 \end{array} \right. , \qquad p, q \in \frac{1}{2}\mathbb{Z}, \qquad i, j = 1, 2, \ldots, N,$$

an action of $\mathcal{L}$ in $\mathcal{H}_{\rm NS}^{\rm fin}$ is defined by the formulæ

$$\begin{aligned} J_m^\alpha \longmapsto J_m^{{\rm NS},\alpha} &= \frac{1}{2} \sum_{i,j=1}^N (T^\alpha)_{i,j} \sum_{r \in \mathbb{Z} + \frac{1}{2}} : b_r^i b_{m-r}^j :, \\ L_m \longmapsto L_m^{\rm NS} &= -\frac{1}{2} \sum_{i=1}^N \sum_{r \in \mathbb{Z} + \frac{1}{2}} \left(r - \frac{m}{2} \right) : b_r^i b_{m-r}^i :, \end{aligned}$$

and in $\mathcal{H}_{\rm R}^{\rm fin}$ by

$$\begin{aligned} J_m^\alpha \longmapsto J_m^{{\rm R},\alpha} &= \frac{1}{2} \sum_{i,j=1}^N (T^\alpha)_{i,j} \sum_{n \in \mathbb{Z}} : b_n^i b_{m-n}^j :, \\ L_m \longmapsto L_m^{\rm R} &= -\frac{1}{2} \sum_{i=1}^N \sum_{n \in \mathbb{Z}} \left(n - \frac{m}{2} \right) : b_n^i b_{m-n}^i : + \frac{N}{16} \delta_{m,0}. \end{aligned}$$

Clearly, these infinite series do not converge in norm, however, using CAR of the b_p^i, they reduce to finite sums when acting on finite particle vectors (18) and (19). Thus $J_m^{{\rm NS},\alpha}, L_n^{\rm NS}$ and $J_m^{{\rm R},\alpha}, L_n^{\rm R}$ are indeed well-defined on $\mathcal{H}_{\rm NS}^{\rm fin}$ and $\mathcal{H}_{\rm R}^{\rm fin}$, respectively. Relations (2), (3) and (4) follow also by direct computation. In $\mathcal{H}_{\rm NS}$ we have states $|0\rangle_{\rm NS} = |\Omega_{\rm NS}\rangle$ and $|i\rangle_{\rm NS} = b_{-\frac{1}{2}}^i |\Omega_{\rm NS}\rangle$, $i = 1, 2, \ldots, N$, which are eigenvectors of $L_0^{\rm NS}$,

$$L_0^{\rm NS}|0\rangle_{\rm NS} = 0, \qquad L_0^{\rm NS}|i\rangle_{\rm NS} = \frac{1}{2}|i\rangle_{\rm NS}.$$

In $\mathcal{H}_{\rm R}$ we have 2^N independent states $|0\rangle_{\rm R} = |\Omega_{\rm R}\rangle$ and

$$|i_l, \dots, i_2, i_1\rangle_{\rm R} = b_0^{i_l} \cdots b_0^{i_2} b_0^{i_1} |\Omega_{\rm R}\rangle, \qquad 1 \le i_1 < i_2 < \cdots < i_l \le N,$$

satisfying

$$L_0^{\rm R}|0\rangle_{\rm R} = \frac{N}{16}|0\rangle_{\rm R}, \qquad L_0^{\rm R}|i_l, \dots, i_2, i_1\rangle_{\rm R} = \frac{N}{16}|i_l, \dots, i_2, i_1\rangle_{\rm R}.$$

As $\mathcal{L}$-modules $\mathcal{H}_{\rm NS}^{\rm fin}$ and $\mathcal{H}_{\rm R}^{\rm fin}$ are not irreducible. It is known that $\mathcal{H}_{\rm NS}^{\rm fin}$ splits up into the direct sum of the basic and the vector module, while $\mathcal{H}_{\rm R}^{\rm fin}$ decomposes into the direct sum of $2^{\frac{N}{2}}$ spinor (s) and $2^{\frac{N}{2}}$ conjugate spinor (c) modules if N is even and into $2^{\frac{N+1}{2}}$ spinor modules (σ) if N is odd. Using our previous results of CAR theory, we can easily verify that exactly the same happens if we restrict the representations $\pi_{\rm NS}$ ($\pi_{\rm R}$) of $\mathcal{C}(\mathcal{K},\Gamma)$ in $\mathcal{H}_{\rm NS}$ ($\mathcal{H}_{\rm R}$) to the even subalgebra $\mathcal{C}(\mathcal{K},\Gamma)^+$: Since $P_{\rm NS}$ is a basis projection we have by Theorem 3.5

$$\pi_{\rm NS}|_{\mathcal{C}(\mathcal{K},\Gamma)^+} = \pi_{\rm NS}^+ \oplus \pi_{\rm NS}^-. \tag{20}$$

Now $\pi_{\rm NS}^+$ acts in the even Fock space [2] which corresponds to the basic module. Thus we may use the same symbols which label the sectors, $\pi_0 \equiv \pi_{\rm NS}^+$ (π_0 being the basic, i.e. vacuum representation) and $\pi_{\rm v} \equiv \pi_{\rm NS}^-$. Consider the Bogoliubov operator $V_{1/2} \in \mathcal{I}(\mathcal{K},\Gamma)$,

$$V_{1/2} = \sum_{i=1}^N \left(\frac{1}{\sqrt{2}}|e^i_{\frac12}\rangle\langle e^i_0| + \frac{1}{\sqrt{2}}|e^i_{-\frac12}\rangle\langle e^i_0| + \sum_{n=1}^\infty \left(|e^i_{n+\frac12}\rangle\langle e^i_n| + |e^i_{-n-\frac12}\rangle\langle e^i_{-n}| \right) \right).$$

It is not hard to see that $S_{\rm R} = V_{1/2}^* P_{\rm NS} V_{1/2}$, that $M_{V_{1/2}} = N$ and that $N_{V_{1/2}} = 0$. We find by Eq. (7) $\pi_{\rm R} \simeq \pi_{\rm NS} \circ \varrho_{V_{1/2}}$ by Eq. (7), and hence by Theorem 3.7,

$$\pi_{\rm R}|_{\mathcal{C}(\mathcal{K},\Gamma)^+} \simeq \begin{cases} 2^{\frac{N}{2}}(\pi_{P'}^+ \oplus \pi_{P'}^-) & N \in 2\mathbb{N} \\ 2^{\frac{N+1}{2}}\pi & N \in 2\mathbb{N}_0 + 1 \end{cases} \tag{21}$$

for a basis projection P', $[P']_2 = [S_{\rm R}^{\frac12}]_2$. Thus we use notations $\pi_{\rm s} \equiv \pi_{P'}^+$, $\pi_{\rm c} \equiv \pi_{P'}^-$ and $\pi_\sigma \equiv \pi$. (Recall that π is one of the equivalent restrictions of the pseudo Fock representations $\pi_{E,\pm}$.) We have seen that the CAR representations $\pi_{\rm NS}$ and $\pi_{\rm R}$, when restricted to the even algebra, reproduce precisely the sectors of the chiral algebra. This is not a surprise because the Kac-Moody and Virasoro generators are made of fermion bilinears. Note that the Bogoliubov endomorphism $\varrho_{V_{1/2}}$ induces a transition from the vacuum sector to spinor sectors.

Let us finish this section with some brief remarks on Möbius covariance of the vacuum sector. The Möbius symmetry on the circle S^1 is given by the group ${\rm PSU}(1,1) = {\rm SU}(1,1)/\mathbb{Z}_2$ where

$${\rm SU}(1,1) = \left\{ g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in {\rm GL}_2(\mathbb{C}) \;\middle|\; |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

Its action on the circle is

$$gz = \frac{\overline{\alpha}z - \overline{\beta}}{-\beta z + \alpha}, \qquad z \in S^1.$$

Consider the one-parameter-group of rotations $a_0(t)$,

$$a_0(t) = \begin{pmatrix} e^{-\frac{it}{2}} & 0 \\ 0 & e^{\frac{it}{2}} \end{pmatrix}, \qquad t \in \mathbb{R}.$$

Any element $g \in \mathrm{SU}(1,1)$ can be decomposed in a rotation $a_0(t)$ and a transformation $g' = a_0(-t)g$ leaving the point $z = -1$ invariant,

$$g = a_0(t)g', \qquad g' = \begin{pmatrix} \alpha' & \beta' \\ \overline{\beta'} & \overline{\alpha'} \end{pmatrix}, \qquad \frac{\overline{\alpha'} + \overline{\beta'}}{\alpha' + \beta'} = 1.$$

Since $a_0(t + 2\pi) = -a_0(t)$ we can determine t, $-2\pi < t \leq 2\pi$ uniquely by the additional requirement $\mathrm{Re}(\alpha') > 0$. Then a representation U of $\mathrm{SU}(1,1)$ in our Hilbert space $\mathcal{K}$ of test functions $f = (f^i)_{i=1,\ldots,N}$ is defined component-wise by

$$\left(U(g)f\right)^i(z) = \epsilon(g;z)(\alpha + \overline{\beta}\overline{z})^{-\frac{1}{2}}(\overline{\alpha} + \beta z)^{-\frac{1}{2}} f^i\left(\frac{\alpha z + \overline{\beta}}{\beta z + \overline{\alpha}}\right)$$

where for $z = e^{i\phi}$, $-\pi < \phi \leq \pi$

$$\epsilon(g;z) = -\,\mathrm{sign}(t - \pi - \phi)\,\mathrm{sign}(t + \pi - \phi),$$

and $\mathrm{sign}(x) = 1$ if $x \geq 0$, $\mathrm{sign}(x) = -1$ if $x < 0$. By the same arguments as in the appendix of [5] for the case $N = 1$ one checks that U is indeed a well-defined and unitary representation. Moreover, since the prefactor on the right hand side is real we observe $[U(g), \Gamma] = 0$ and hence each $U(g)$, $g \in \mathrm{SU}(1,1)$, induces a Bogoliubov automorphism $\alpha_g = \varrho_{U(g)}$. It follows that $\mathrm{SU}(1,1)$ is represented by automorphisms of $\mathcal{C}(\mathcal{K},\Gamma)$, and this restricts to a representation of $\mathrm{PSU}(1,1)$ by automorphisms of $\mathcal{C}(\mathcal{K},\Gamma)^+$. Again, as an obvious generalization of the computations in [5] one checks $[P_{\mathrm{NS}}, U(g)] = 0$ and hence $\pi_{\mathrm{NS}} \circ \alpha_g \simeq \pi_{\mathrm{NS}}$ and $\pi_0 \circ \alpha_g \simeq \pi_0$. Consider further one-parameter-subgroups

$$a_+(t) = \begin{pmatrix} \cosh t & i \sinh t \\ -i \sinh t & \cosh t \end{pmatrix}, \qquad a_-(t) = \begin{pmatrix} \cosh t & \sinh t \\ -\sinh t & \cosh t \end{pmatrix},$$

$t \in \mathbb{R}$. It is not hard to check that the a_ϵ correspond to infinitesimal generators d_0, $d_+ = d_1 + d_{-1}$ and $d_- = -i(d_1 - d_{-1})$ by $U(a_\epsilon(t)) = \exp(itd_\epsilon)$, $\epsilon = 0, \pm$, where

$$d_n = -\sum_{i=1}^{N} \sum_{r \in \mathbb{Z}+\frac{1}{2}} \left(r + \frac{n}{2}\right) |e^i_{r+n}\rangle\langle e^i_r|, \qquad n = 0, \pm 1,$$

i.e. the d_n act as $-z^n\left(z\frac{d}{dz} + \frac{n}{2}\right)$ in each component. The relation

$$[L^{\mathrm{NS}}_n, b^i_r] = -\left(r + \frac{n}{2}\right) b^i_{r+n}, \qquad r \in \mathbb{Z} + \frac{1}{2}, \quad i = 1, 2, \ldots, N, \quad n = 0, \pm 1,$$

establishes the correspondence between generators d_ϵ in the test function space $\mathcal{K}$ and infinitesimal generators L^{NS}_ϵ in the Fock space $\mathcal{H}_{\mathrm{NS}}$; we have (for f in the domain of d_ϵ)

$$[L^{\mathrm{NS}}_\epsilon, \pi_{\mathrm{NS}}(B(f))] = \pi_{\mathrm{NS}}(B(d_\epsilon f)), \qquad \epsilon = 0, \pm,$$

where $L^{\mathrm{NS}}_+ = L^{\mathrm{NS}}_1 + L^{\mathrm{NS}}_{-1}$ and $L^{\mathrm{NS}}_- = -i(L^{\mathrm{NS}}_1 - L^{\mathrm{NS}}_{-1})$.

5 Localized Endomorphisms

In this section we construct localized endomorphisms by means of Bogoliubov endomorphisms. Thus we have to introduce at first a local structure on S^1, i.e. to define local algebras of observables. Let us denote by $\mathcal{J}$ the set of open, non-void proper subintervals of S^1. For $I \in \mathcal{J}$ set $\mathcal{K}(I) = L^2(I;\mathbb{C}^N)$ and define local C^*-algebras

$$\mathcal{A}(I) = \mathcal{C}(\mathcal{K}(I), \Gamma)^+$$

such that we have inclusions

$$\mathcal{A}(I) \subset \mathcal{A}(I_0), \qquad I \subset I_0,$$

inherited by the natural embedding of the L^2-spaces; and also we have locality,

$$[\mathcal{A}(I), \mathcal{A}(I_1)] = \{0\}, \qquad I \cap I_1 = \emptyset.$$

Our construction of localized endomorphisms happens on the punctured circle. Consider the interval $I_\zeta \in \mathcal{J}$ which is S^1 by removing one "point at infinity" $\zeta \in S^1$, $I_\zeta = S^1 \setminus \{\zeta\}$. Clearly, $\mathcal{A}(I_\zeta) = \mathcal{C}(\mathcal{K}, \Gamma)^+$. Further denote by $\mathcal{J}_\zeta$ the set of "finite" intervals $I \in \mathcal{J}$ such that their closure is contained in I_ζ,

$$\mathcal{J}_\zeta = \{ I \in \mathcal{J} \mid \bar{I} \subset I_\zeta \}.$$

An endomorphism ϱ of $\mathcal{A}(I_\zeta)$ is called localized in some interval $I \in \mathcal{J}_\zeta$ if it satisfies

$$\varrho(A) = A, \qquad A \in \mathcal{A}(I_1), \qquad I_1 \in \mathcal{J}_\zeta, \qquad I_1 \cap I = \emptyset.$$

The construction of localized endomorphisms by means of Bogoliubov transformations leads to the concept of pseudo-localized isometries [17]. For $I \in \mathcal{J}_\zeta$ denote by I_+ and I_- the two connected components of $I' \cap I_\zeta$ (I' always denotes the interior of the complement of I in S^1, $I' = I^{\mathrm{c}} \setminus \partial I^{\mathrm{c}}$). A Bogoliubov operator $V \in \mathcal{I}(\mathcal{K}, \Gamma)$ is called even (resp. odd) pseudo-localized in $I \in \mathcal{J}_\zeta$ if

$$Vf = \epsilon_\pm f, \qquad f \in \mathcal{K}(I_\pm), \qquad \epsilon_\pm \in \{-1, 1\},$$

and $\epsilon_+ = \epsilon_-$ (resp. $\epsilon_+ = -\epsilon_-$). Then, as obvious, ϱ_V is localized in I in restriction to $\mathcal{A}(I_\zeta)$. Now we are ready to define our localized vector endomorphism.

Definition 5.1 *For some $I \in \mathcal{J}_\zeta$ choose a real $v \in \mathcal{K}(I)$, $\Gamma v = v$ and $\|v\| = 1$. Define the unitary self-adjoint Bogoliubov operator $U \in \mathcal{I}(\mathcal{K}, \Gamma)$ by*

$$U = 2|v\rangle\langle v| - \mathbf{1}, \tag{22}$$

and the localized vector endomorphism (automorphism) ϱ_{v} by $\varrho_{\mathrm{v}} = \varrho_U$.

Since U is even pseudo-localized, and by Corollary 3.6, ϱ_{v} is indeed a localized vector endomorphism, i.e. $\pi_0 \circ \varrho_{\mathrm{v}} \simeq \pi_{\mathrm{v}}$. Further, by $U^2 = \mathbf{1}$ we have $\pi_0 \circ \varrho_{\mathrm{v}}^2 \simeq \pi_0$. It follows also by Corollary 3.6 that $\pi_{\mathrm{s}} \circ \varrho_{\mathrm{v}} \simeq \pi_{\mathrm{c}}$. The construction of a localized spinor endomorphism is a little bit more costly. Without loss of generality, we choose $\zeta = -1$ and the localization region to be I_2,

$$I_2 = \left\{ z = \mathrm{e}^{\mathrm{i}\phi} \in S^1 \;\middle|\; -\frac{\pi}{2} < \phi < \frac{\pi}{2} \right\}$$

such that the connected components $I_\pm$ of $I_2' \cap I_\zeta$ are given by

$$I_- = \left\{ z = \mathrm{e}^{\mathrm{i}\phi} \in S^1 \;\middle|\; -\pi < \phi < -\frac{\pi}{2} \right\}, \quad I_+ = \left\{ z = \mathrm{e}^{\mathrm{i}\phi} \in S^1 \;\middle|\; \frac{\pi}{2} < \phi < \pi \right\}.$$

Our Hilbert space $\mathcal{K} = \mathcal{K}(I_\zeta)$ decomposes into a direct sum,

$$\mathcal{K} = \mathcal{K}(I_-) \oplus \mathcal{K}(I_2) \oplus \mathcal{K}(I_+).$$

By P_{I_+}, P_{I_-} we denote the projections onto the subspaces $\mathcal{K}(I_+)$, $\mathcal{K}(I_-)$, respectively. Define functions on S^1 by

$$f_p(z) = \begin{cases} \sqrt{2} z^{2p} & z \in I_2 \\ 0 & z \notin I_2 \end{cases}, \qquad p \in \frac{1}{2}\mathbb{Z},$$

and

$$f_p^i = f_p \otimes u^i, \qquad p \in \frac{1}{2}\mathbb{Z}, \qquad i = 1, 2, \ldots, N,$$

such that we obtain two ONB of the subspace $\mathcal{K}(I_2) \subset \mathcal{K}$,

$$\left\{ f_r^i, r \in \mathbb{Z} + \frac{1}{2}, i = 1, 2, \ldots, N \right\}, \qquad \{ f_n^i, n \in \mathbb{Z}, i = 1, 2, \ldots, N \}.$$

Now define the odd pseudo-localized Bogoliubov operator $V \in \mathcal{I}(\mathcal{K}, \Gamma)$,

$$\begin{aligned}
V &= P_{I_-} - P_{I_+} + V^{(2)}, && (23) \\
V^{(2)} &= \sum_{\substack{j \le N \\ j \text{ odd}}} (\mathrm{i} r^j + \mathrm{i} R^j) - \sum_{\substack{j \le N \\ j \text{ even}}} (t^j + \mathrm{i} T^j), && (24) \\
r^j &= \frac{1}{\sqrt{2}} |f^j_{\frac{1}{2}}\rangle\langle f^j_0| - \frac{1}{\sqrt{2}} |f^j_{-\frac{1}{2}}\rangle\langle f^j_0|, && (25) \\
R^j &= \sum_{n=1}^{\infty} \Big(|f^j_{n+\frac{1}{2}}\rangle\langle f^j_n| - |f^j_{-n-\frac{1}{2}}\rangle\langle f^j_{-n}| \Big), && (26) \\
t^j &= \frac{1}{\sqrt{2}} |f^{j-1}_{\frac{1}{2}}\rangle\langle f^j_0| + \frac{1}{\sqrt{2}} |f^{j-1}_{-\frac{1}{2}}\rangle\langle f^j_0|, && (27) \\
T^j &= \sum_{n=1}^{\infty} \Big(|f^j_{n-\frac{1}{2}}\rangle\langle f^j_n| - |f^j_{-n+\frac{1}{2}}\rangle\langle f^j_{-n}| \Big). && (28)
\end{aligned}$$

Remark that V is unitary if $N \in 2\mathbb{N}$. In particular we have

$$M_V = \begin{cases} 0 & N \in 2\mathbb{N} \\ 1 & N \in 2\mathbb{N}_0 + 1 \end{cases}.$$

Moreover, we claim

Lemma 5.2 *With notations as above,*

$$\begin{aligned}
[(V^* P_{\mathrm{NS}} V)^{\frac{1}{2}}]_2 &= [S_{\mathrm{R}}^{\frac{1}{2}}]_2, && (29) \\
[(V^* V^* P_{\mathrm{NS}} V V)^{\frac{1}{2}}]_2 &= [P_{\mathrm{NS}}]_2. && (30)
\end{aligned}$$

Proof. Let us first point out that that we do not have to take care about the positive square roots because for any basis projection P and any Bogoliubov operator $W \in \mathcal{I}(\mathcal{K},\Gamma)$ with $M_W < \infty$ we have

$$[(W^*PW)^{\frac{1}{2}}]_2 = [W^*PW]_2$$

since

$$\begin{aligned} \|(W^*PW)^{\frac{1}{2}} - W^*PW\|_2^2 &\leq \|W^*PW - (W^*PW)^2\|_1 \\ &= \|W^*P(\mathbf{1} - WW^*)PW\|_1 \\ &\leq \|W\|^2\|P\|^2\|\mathbf{1} - WW^*\|_1 = M_W. \end{aligned}$$

Here we used the trace norm and Hilbert Schmidt norm $\|A\|_n = (\mathrm{tr}(A^*A)^{\frac{n}{2}})^{\frac{1}{n}}$, $n = 1,2$, respectively, and also an estimate [18]

$$\|A^{\frac{1}{2}} - B^{\frac{1}{2}}\|_2^2 \leq \|A - B\|_1, \qquad A,B \in \mathfrak{B}(\mathcal{K}), \qquad A,B \geq 0. \tag{31}$$

It was proven in [5], Lemma 3.10, that

$$V^*P_{\mathrm{NS}}V - S_{\mathrm{R}}, \quad VP_{\mathrm{NS}}V^* - S_{\mathrm{R}}, \quad {V'}^*P_{\mathrm{NS}}V' - S_{\mathrm{R}}, \quad V'P_{\mathrm{NS}}{V'}^* - S_{\mathrm{R}}$$

are Hilbert Schmidt operators for the case $N = 1$, where in our notation

$$V = P_{I_-} - P_{I_+} + \mathrm{i}r^1 + \mathrm{i}R^1, \qquad V' = P_{I_-} - P_{I_+} + \mathrm{i}(T^1)^*.$$

For arbitrary N operators $V^*P_{\mathrm{NS}}V - S_{\mathrm{R}}$ and $VP_{\mathrm{NS}}V^* - S_{\mathrm{R}}$ are just direct sums of the above Hilbert Schmidt operators (up to finite dimensional operators), hence we conclude for arbitrary N

$$V^*P_{\mathrm{NS}}V - S_{\mathrm{R}} \in \mathfrak{J}_2(\mathcal{K}), \qquad VP_{\mathrm{NS}}V^* - S_{\mathrm{R}} \in \mathfrak{J}_2(\mathcal{K}).$$

Both relations together imply that $P_{\mathrm{NS}} - V^*V^*P_{\mathrm{NS}}VV$ is also Hilbert Schmidt which proves the lemma, q.e.d.

Hence we conclude $\pi_{\mathrm{NS}} \circ \varrho_V \approx \pi_{\mathrm{R}}$. For $N \in 2\mathbb{N}$ the basis projection $P' = V^*P_{\mathrm{NS}}V$ is as in Eq. (21). For $N \in 2\mathbb{N}_0 + 1$ the representation $\pi_{\mathrm{NS}} \circ \varrho_V$, when restricted to $\mathcal{C}(\mathcal{K},\Gamma)^+$, decomposes into two equivalent irreducibles. With our above definitions and using Corollary 3.6, this suggests the following

Definition 5.3 *Choose $U \in \mathcal{I}(\mathcal{K},\Gamma)$ for $v \in \mathcal{K}(I_2)$ as in Definition 5.1. For $N \in 2\mathbb{N}$ define the localized spinor endomorphism ϱ_{s} by $\varrho_{\mathrm{s}} = \varrho_V$ and the localized conjugate spinor endomorphism ϱ_{c} by $\varrho_{\mathrm{c}} = \varrho_U\varrho_V$. For $N \in 2\mathbb{N}_0 + 1$ define the localized spinor endomorphism ϱ_σ by $\varrho_\sigma = \varrho_V$.*

Note that this definition fixes the choice, if N is even, which of the two inequivalent spinor sectors is called s and which c. There is no loss of generality because the fusion rules turn out to be invariant under exchange of s and c. Indeed, our considerations have shown

Theorem 5.4 *The localized endomorphisms of Definitions 5.1 and 5.3 satisfy $\pi_0 \circ \varrho_J \simeq \pi_J$, $J = \mathrm{v},\mathrm{s},\mathrm{c},\sigma$.*

6 Extension to Local von Neumann Algebras

We have obtained the relevant localized endomorphisms which generate the sectors $\mathrm{v}, \mathrm{s}, \mathrm{c}, \sigma$. It is our next aim to derive fusion rules in terms of DHR sectors i.e. of unitary equivalence classes $[\pi_0 \circ \varrho]$ for localized endomorphisms ϱ. For such a formulation one needs local intertwiners in the observable algebra. So we have to keep close to the DHR framework, in particular, we should use local von Neumann algebras instead of local C^*-algebras $\mathcal{A}(I)$. We define

$$\mathcal{R}(I) = \pi_0(\mathcal{A}(I))'', \qquad I \in \mathcal{J}.$$

By Möbius covariance of the vacuum state, this defines a so-called covariant precosheaf on the circle [7]. In particular, we have Haag duality,

$$\mathcal{R}(I)' = \mathcal{R}(I'). \tag{32}$$

Since the set $\mathcal{J}$ is not directed by inclusion we cannot define a global algebra as the C^*-norm closure of the union of all local algebras. However, the set $\mathcal{J}_\zeta$ is directed so that we can define the following algebra $\mathfrak{A}_{\mathrm{loc}}$ of quasilocal observables in the usual manner,

$$\mathfrak{A}_{\mathrm{loc}} = \overline{\bigcup_{I \in \mathcal{J}_\zeta} \mathcal{R}(I)}. \tag{33}$$

We want to prove that Haag duality holds also on the punctured circle and need some technical preparation. Recall that a function $k \in L^2(S^1)$ is in the Hardy space H^2 if $\langle e_{-n}, k \rangle = 0$ for all $n \in \mathbb{N}$ where $e_{-n}(z) = z^{-n}$. There is a Theorem of Riesz ([10], Th. 6.13) which states that $k(z) \neq 0$ almost everywhere if $k \in H^2$ is non-zero. Now suppose $f \in P_{\mathrm{NS}}\mathcal{K}$. Then $g^i \in H^2$ where $g^i(z) = z^{\frac{1}{2}}\overline{f^i(z)}$ component-wise, $i = 1, 2, \ldots, N$. We conclude

Lemma 6.1 *If $f \in P_{\mathrm{NS}}\mathcal{K}$ then $f \in \mathcal{K}(I)$ implies $f = 0$ for any $I \in \mathcal{J}$.*

For some interval $I \in \mathcal{J}_\zeta$, let us denote by $\mathfrak{A}_\zeta(I')$ the norm closure of the algebra generated by all $\mathcal{R}(I_1)$, $I_1 \in \mathcal{J}_\zeta$, $I_1 \cap I = \emptyset$. Obviously $\mathfrak{A}_\zeta(I')'' \subset \mathcal{R}(I')$; a key point of the analysis is the following

Lemma 6.2 *Haag duality remains valid on the punctured circle, i.e.*

$$\mathcal{R}(I)' = \mathfrak{A}_\zeta(I')''. \tag{34}$$

Proof. We have to prove $\mathfrak{A}_\zeta(I')'' = \mathcal{R}(I')$. It is sufficient to show that each generator $\pi_0(B(f)B(g))$, $f, g \in \mathcal{K}(I')$ of $\mathcal{R}(I')$ is a weak limit point of a net in $\mathfrak{A}_\zeta(I')$. Note that the subspace $\mathcal{K}^{(\zeta)}(I') \subset \mathcal{K}(I')$ of functions which vanish in a neighborhood of ζ is dense. So by Eq. (5) we conclude that it is sufficient to establish this fact only for such generators with $f, g \in \mathcal{K}^{(\zeta)}(I')$, because these generators approximate the arbitrary ones already in the norm topology. Let us again denote the two connected components of $I' \setminus \{\zeta\}$ by I_+ and I_-, and the projections onto corresponding subspaces $\mathcal{K}(I_\pm)$ by $P_\pm$. We also write $f_\pm = P_\pm f$ and $g_\pm = P_\pm g$ for our functions $f, g \in \mathcal{K}^{(\zeta)}(I')$. Then we have

$$\begin{aligned} \pi_0(B(f)B(g)) &= \pi_0(B(f_+)B(g_+)) + \pi_0(B(f_-)B(g_-)) \\ &\quad + \pi_0(B(f_+)B(g_-)) + \pi_0(B(f_-)B(g_+)). \end{aligned}$$

Clearly, the first two terms on the r.h.s. are elements of $\mathfrak{A}_\zeta(I')$. We show that the third term $Y = \pi_0(B(f_+)B(g_-))$ (then, by symmetry, also the fourth one) is in $\mathfrak{A}_\zeta(I')''$. In the same way as in the proof of Lemma 4.1 in [5] one constructs a sequence $\{X_n, n \in \mathbb{N}\}$,

$$X_n = \pi_0(B(h_n^+)B(h_n^-))$$

where unit vectors $h_n^\pm \in \mathcal{K}(I_n^\pm)$ are related by Möbius transformations such that intervals $I_n^\pm \subset I_\pm$ shrink to the point ζ. Since $\|X_n\| \le 1$ by Eq. (5) it follows that there is a weakly convergent subnet $\{Z_\alpha, \alpha \in \iota\}$ (ι a directed set), $\mathrm{w}\!-\lim_\alpha Z_\alpha = Z$. For each $I_0 \in \mathcal{J}_\zeta$ elements X_n commute with each $A \in \mathcal{R}(I_0)$ for sufficiently large n. Hence Z is in the commutant of $\mathfrak{A}_{\mathrm{loc}}$ and this implies $Z = \lambda\mathbf{1}$. We have chosen the vectors $h_n^\pm$ related by Möbius transformations. By Möbius invariance of the vacuum state we have

$$\lambda = \langle \Omega_0 | X_1 | \Omega_0 \rangle = \langle \Gamma h_1^+, P_{\mathrm{NS}} h_1^- \rangle.$$

We claim that we can choose $h_1^\pm$ such that $\lambda \neq 0$. For given h_1^- set $k = P_{\mathrm{NS}} h_1^-$. We have $k \neq 0$, otherwise $\Gamma h_1^- \in P_{\mathrm{NS}}\mathcal{K}$ in contradiction to $h_1^- \in \mathcal{K}(I_1^-)$ by Lemma 6.1. Again by Lemma 6.1 we conclude that k cannot vanish almost everywhere. So we clearly can choose a $h_1^+ \in \mathcal{K}(I_1^+)$ such that $\lambda = \langle \Gamma h_1^+, k\rangle \neq 0$. Now we find $Y = \lambda^{-1}\mathrm{w}\!-\lim_\alpha YZ_\alpha$ and also $YZ_\alpha \in \mathfrak{A}_\zeta(I')$ because

$$YX_n = \pi_0(B(f_+)B(g_-)B(h_n^+)B(h_n^-)) = -\pi_0(B(f_+)B(h_n^+))\pi_0(B(g_-)B(h_n^-))$$

is in $\mathfrak{A}_\zeta(I')$ for all $n \in \mathbb{N}$, q.e.d.

Since the vacuum representation is faithful on $\mathcal{A}(I_\zeta)$ we can identify observables A in the usual manner with their vacuum representers $\pi_0(A)$. Thus we consider the vacuum representation as acting as the identity on $\mathfrak{A}_{\mathrm{loc}}$, and, in the same fashion, we treat local C^*-algebras as subalgebras $\mathcal{A}(I) \subset \mathcal{R}(I)$. Now we have to check whether we can extend our representations π_J and endomorphisms ϱ_J from $\mathcal{A}(I)$ to $\mathcal{R}(I) = \mathcal{A}(I)''$, $I \in \mathcal{J}_\zeta$, $J = \mathrm{v}, \mathrm{s}, \mathrm{c}, \sigma$. That is that we have to check local quasiequivalence of the representations π_J and will now be elaborated. Define $E_{\mathrm{R}} \in \mathfrak{B}(\mathcal{K})$ by

$$E_{\mathrm{R}} = \sum_{i=1}^{N} \sum_{n \in \mathbb{N}} |e_{-n}^i\rangle\langle e_{-n}^i| + \sum_{\substack{j \le N \\ j \text{ even}}} |e_+^j\rangle\langle e_+^j|$$

where $e_+^j = 2^{-\frac{1}{2}}(e_0^j + \mathrm{i} e_0^{j-1})$.

Lemma 6.3 *For $I \in \mathcal{J}$ the subspaces $P_{\mathrm{NS}}\mathcal{K}(I) \subset P_{\mathrm{NS}}\mathcal{K}$ and $E_{\mathrm{R}}\mathcal{K}(I) \subset E_{\mathrm{R}}\mathcal{K}$ are dense.*

Proof. Suppose that $P_{\mathrm{NS}}\mathcal{K}(I)$ is not dense in $P_{\mathrm{NS}}\mathcal{K}$. Then there is a non-zero $f \in P_{\mathrm{NS}}\mathcal{K}$ such that

$$\langle f, P_{\mathrm{NS}} g\rangle = \langle f, g\rangle = 0$$

for all $g \in \mathcal{K}(I)$. Hence $f \in \mathcal{K}(I)^\perp = \mathcal{K}(I')$ in contradiction to Lemma 6.1. As quite obvious, Lemma 6.1 holds for $f \in E_{\mathrm{R}}\mathcal{K}$ as well. So also $E_{\mathrm{R}}\mathcal{K}(I)$ is dense in $E_{\mathrm{R}}\mathcal{K}$, q.e.d.

Note that $E_{\rm R}$ is a basis projection if N is even. For N odd, $E_{\rm R}$ is a partial basis projection with Γ-codimension 1 and corresponding Γ-invariant unit vector e_0^N. In this case

$$S'_{\rm R} = \frac{1}{2}|e_0^N\rangle\langle e_0^N| + E_{\rm R}$$

is of the form (9). Let us denote by $(\mathcal{H}_{\rm R'}, \pi_{\rm R'}, |\Omega_{\rm R'}\rangle)$ the GNS representation of the quasifree state $\omega_{E_{\rm R}}$ if N is even and $\omega_{S'_{\rm R}}$ if N is odd. We conclude

$$\pi_{\rm R'}|_{\mathcal{C}(\mathcal{K},\Gamma)^+} \simeq \begin{cases} \pi_{\rm s} \oplus \pi_{\rm c} & N \in 2\mathbb{N} \\ 2\pi_\sigma & N \in 2\mathbb{N}_0 + 1 \end{cases}$$

by Theorem 3.5 and Lemma 3.2 and the fact that $[E_{\rm R}]_2 = [S_{\rm R}^{\frac{1}{2}}]_2 = [S_{\rm R}]_2$ (N even) and $[{S'_{\rm R}}^{\frac{1}{2}}]_2 = [S'_{\rm R}]_2 = [S_{\rm R}]_2$ (N odd).

Lemma 6.4 *For $I \in \mathcal{J}_\zeta$ we have local quasiequivalence*

$$\pi_{\rm NS}|_{\mathcal{C}(\mathcal{K}(I),\Gamma)} \approx \pi_{\rm R'}|_{\mathcal{C}(\mathcal{K}(I),\Gamma)}. \tag{35}$$

Proof. We first claim that $|\Omega_{\rm NS}\rangle$ and $|\Omega_{\rm R'}\rangle$ remain cyclic for $\pi_{\rm NS}(\mathcal{C}(\mathcal{K}(I),\Gamma))$ and $\pi_{\rm R'}(\mathcal{C}(\mathcal{K}(I),\Gamma))$, respectively. By Lemma 6.3, $P_{\rm NS}\mathcal{K}(I) \subset P_{\rm NS}\mathcal{K}$ is dense. Hence vectors $\pi_{\rm NS}(B(f_1)\cdots B(f_n))|\Omega_{\rm NS}\rangle$, with $f_1,\ldots,f_n \in P_{\rm NS}\mathcal{K}(I)$, $n = 0,1,2,\ldots$, are total in $\mathcal{H}_{\rm NS}$. This proves the required cyclicity of $|\Omega_{\rm NS}\rangle$. For N even, cyclicity of $|\Omega_{\rm R'}\rangle$ for $\mathcal{C}(\mathcal{K}(I),\Gamma)$ is proven in the same way. For N odd, we have $\mathcal{H}_{\rm R'} = \mathcal{H}_{E_{\rm R}} \oplus \mathcal{H}_{E_{\rm R}}$, $\pi_{\rm R'} = \pi_{E,+} \oplus \pi_{E,-}$ and $|\Omega_{\rm R'}\rangle = 2^{-\frac{1}{2}}(|\Omega_{E_{\rm R}}\rangle \oplus |\Omega_{E_{\rm R}}\rangle)$ as in Lemma 3.2, and the corresponding Γ-invariant unit vector is given by e_0^N. In order to prove cyclicity of $|\Omega_{\rm R'}\rangle$ we show that $\langle\Psi|\pi_{\rm R'}(x)|\Omega_{\rm R'}\rangle = 0$ for all $x \in \mathcal{C}(\mathcal{K}(I),\Gamma)$, $|\Psi\rangle = |\Psi_+\rangle \oplus |\Psi_-\rangle \in \mathcal{H}_{\rm R'}$, implies $|\Psi\rangle = 0$. We have

$$\langle\Psi|\pi_{\rm R'}(x)|\Omega_{\rm R'}\rangle = \frac{1}{\sqrt{2}}\langle\Psi_+|\pi_{E_{\rm R},+}(x)|\Omega_{E_{\rm R}}\rangle + \frac{1}{\sqrt{2}}\langle\Psi_-|\pi_{E_{\rm R},-}(x)|\Omega_{E_{\rm R}}\rangle = 0$$

Again by Lemma 6.3, $E_{\rm R}\mathcal{K}(I) \subset E_{\rm R}\mathcal{K}$ is dense, hence vectors $\pi_{E_{\rm R},\pm}(x)|\Omega_{E_{\rm R}}\rangle = \pi_{E_{\rm R}}(x)|\Omega_{E_{\rm R}}\rangle$, $x = B(f_1)\cdots B(f_n)$, $f_1,\ldots,f_n \in E_{\rm R}\mathcal{K}(I)$, $n = 0,1,2,\ldots$, are total in $\mathcal{H}_{E_{\rm R}}$. It follows $|\Psi_-\rangle = -|\Psi_+\rangle$. Hence

$$\langle\Psi_+|(\pi_{E_{\rm R},+}(y) - \pi_{E_{\rm R},-}(y))|\Omega_{E_{\rm R}}\rangle = 0, \qquad y \in \mathcal{C}(\mathcal{K}(I),\Gamma).$$

Keep all $x = B(f_1)\cdots B(f_n)$ as above and choose an $f \in \mathcal{K}(I)$ such that $\langle e_0^N, f\rangle = 2^{-\frac{1}{2}}$. Set $y = (-1)^n B(f)x$. Then, by Eq. (8), we compute

$$\pi_{E_{\rm R},\pm}(y) = (-1)^n \left(\pm\frac{1}{2}Q_{E_{\rm R}}(-1) + \pi_{E_{\rm R}}(B((E_{\rm R} + \overline{E}_{\rm R})f))\right)\pi_{E_{\rm R}}(x)$$

and hence

$$(\pi_{E_{\rm R},+}(y) - \pi_{E_{\rm R},-}(y))|\Omega_{E_{\rm R}}\rangle = \pi_{E_{\rm R}}(B(f_1)\cdots B(f_n)|\Omega_{E_{\rm R}}\rangle.$$

Because such vectors are total in $\mathcal{H}_{E_{\rm R}}$ we find $|\Psi_+\rangle = 0$ and hence $|\Psi\rangle = 0$. We have seen that vectors $|\Omega_{\rm NS}\rangle$ and $|\Omega_{\rm R'}\rangle$ remain cyclic. Thus we can prove the lemma by showing that the restricted states $\omega_{P_I P_{\rm NS} P_I}$ and $\omega_{P_I E_{\rm R} P_I}$ ($N \in 2\mathbb{N}$)

respectively $\omega_{P_I S'_R P_I}$ $(N \in 2\mathbb{N}_0 + 1)$ give rise to quasiequivalent representations. Because they are quasifree on $\mathcal{C}(\mathcal{K}(I), \Gamma)$ we have to show that

$$[(P_I P_{\rm NS} P_I)^{\frac{1}{2}}]_2 = \begin{cases} [(P_I E_{\rm R} P_I)^{\frac{1}{2}}]_2 & N \in 2\mathbb{N} \\ [(P_I S'_{\rm R} P_I)^{\frac{1}{2}}]_2 & N \in 2\mathbb{N}_0 + 1 \end{cases}.$$

By use of Eq. (31) it is sufficient to show that the difference of $P_I P_{\rm NS} P_I$ and $P_I E_{\rm R} P_I$ respectively $P_I S'_{\rm R} P_I$ is trace class. It was proven in [5] that for $I \in \mathcal{J}_\zeta$ the difference $P_I P_{\rm NS} P_I - P_I S_{\rm R} P_I$ is trace class in the case $N = 1$. The result follows because the operators above are, up to finite dimensional operators, direct sums of those for $N = 1$, q.e.d.

In restriction to the local even algebra $\mathcal{A}(I) = \mathcal{C}(\mathcal{K}(I), \Gamma)^+$, $I \in \mathcal{J}_\zeta$ we find by Lemma 6.4

$$(\pi_0 \oplus \pi_{\rm v})|_{\mathcal{A}(I)} \approx \begin{cases} (\pi_{\rm s} \oplus \pi_{\rm c})|_{\mathcal{A}(I)} & N \in 2\mathbb{N} \\ 2\pi_\sigma|_{\mathcal{A}(I)} & N \in 2\mathbb{N}_0 + 1 \end{cases}$$

Recall that $\pi_{\rm v} \simeq \pi_0 \circ \varrho_U$ with $U = 2|v\rangle\langle v| - \mathbf{1}$ as in Corollary 3.6. Choose $v \in \mathcal{K}(I')$. Then $\varrho_U(x) = x$ for $x \in \mathcal{A}(I)$, hence π_0 and $\pi_{\rm v}$ are equivalent on $\mathcal{A}(I)$. In the same way we obtain local equivalence of $\pi_{\rm s}$ and $\pi_{\rm c}$. We conclude

Theorem 6.5 (Local Normality) *In restriction to local C^*-algebras $\mathcal{A}(I)$, $I \in \mathcal{J}_\zeta$, the representations π_J are quasiequivalent to the vacuum representation $\pi_0 = id$,*

$$\pi_J|_{\mathcal{A}(I)} \approx \pi_0|_{\mathcal{A}(I)}, \qquad I \in \mathcal{J}_\zeta, \quad J = {\rm v}, {\rm s}, {\rm c}, \sigma. \tag{36}$$

We have seen that we have an extension of our representations π_J to local von Neumann algebras $\mathcal{R}(I)$, $I \in \mathcal{J}_\zeta$, and thus to the quasilocal algebra $\mathfrak{A}_{\rm loc}$ they generate. By unitary equivalence $\varrho_J \simeq \pi_J$ on $\mathcal{A}(I_\zeta)$ we have an extension of ϱ_J to $\mathfrak{A}_{\rm loc}$, too, $J = {\rm v}, {\rm s}, {\rm c}, \sigma$. Being localized in some $I \in \mathcal{J}_\zeta$, they inherit properties

$$\varrho_J(A) = A, \qquad A \in \mathfrak{A}_\zeta(I'),$$

and

$$\varrho_J(\mathcal{R}(I_0)) \subset \mathcal{R}(I_0), \qquad I_0 \in \mathcal{J}_\zeta, \qquad I \subset I_0,$$

from the underlying C^*-algebras. So our endomorphisms ϱ_J, $J = {\rm v}, {\rm s}, {\rm c}, \sigma$ are well-defined localized endomorphisms $\mathfrak{A}_{\rm loc}$ in the common sense. Moreover, they are transportable. This follows because the precosheaf $\{\mathcal{R}(I)\}$ is Möbius covariant. Hence $\mathfrak{A}_{\rm loc}$ is covariant with respect to the subgroup of Möbius transformations leaving ζ invariant.

7 Fusion Rules

In this section we prove the fusion rules of our sectors $1, {\rm v}, {\rm s}, {\rm c}, \sigma$ in terms of unitary equivalence classes of localized endomorphisms $[\varrho] \equiv [\pi_0 \circ \varrho]$ (or, equivalently, in terms of equivalence classes $[\pi]$ of representations π satisfying an DHR criterion). Because we deal with an Haag dual net of local von Neumann algebras, by standard arguments, it suffices to check a fusion rule $[\varrho_J \varrho_{J'}]$ for special representatives $\varrho_J \in [\varrho_J]$, $\varrho_{J'} \in [\varrho_{J'}]$. This will be done by our examples of Definitions 5.1 and 5.3. Let us denote the unitary equivalence class $[\varrho_J]$ simply by J, the fusion $[\varrho_J \varrho_{J'}]$ by $J * J'$ and the direct sum $[\varrho_J] \oplus [\varrho_{J'}]$ by $J + J'$,

$J, J' = 1, \mathrm{v}, \mathrm{s}, \mathrm{c}, \sigma$. For instance, we clearly have $\mathrm{v} * \mathrm{v} = 1$ for all $N \in \mathbb{N}$. Let us first consider the even case, $N \in 2\mathbb{N}$. By Corollary 3.6 we easily find $\mathrm{v} * \mathrm{s} = \mathrm{c}$, $\mathrm{v} * \mathrm{c} = \mathrm{s}$. Since V then is unitary and by Lemma 5.2 we have $\pi_{\mathrm{NS}} \circ \varrho_V^2 \simeq \pi_{\mathrm{NS}}$. Now π_{NS}, when restricted to $\mathcal{A}(I_\zeta) \equiv \mathcal{C}(\mathcal{K}, \Gamma)^+$, decomposes into the basic and the vector representation. Hence only the possibilities $\mathrm{s} * \mathrm{s} = 1$ or $\mathrm{s} * \mathrm{s} = \mathrm{v}$ are left, i.e. we have to check whether $\pi_{\mathrm{NS}}^+ \circ \varrho_V^2$ is equivalent to π_{NS}^+ or π_{NS}^-, i.e. whether ϱ_{s} is a self-conjugate endomorphism or not. For N even the action of V in the $(2j-1)^{\mathrm{th}}$ and the $2j^{\mathrm{th}}$ component, $j = 1, 2, \ldots, N/2$, is the same as in the 1^{st} and the 2^{nd} component. So we can write the square $W = V^2$ as a product,

$$W = W_{1,2} W_{3,4} \cdots W_{N-1,N}$$

where $W_{1,2}$ acts as W in the first two components and as the identity in the others, etc. Since σ of Prop. 3.4 is multiplicative and clearly all $W_{2j-1,2j}$ lead to implementable automorphisms we have

$$\sigma(W) = \sigma(W_{1,2}) \sigma(W_{3,4}) \cdots \sigma(W_{N-1,N}).$$

All $W_{2j-1,2j}$ are built in the same way, hence all the $\sigma(W_{2j-1,2j})$ are equal i.e. $\sigma(W) = \sigma(W_{1,2})^{N/2}$. Since σ takes only values ± 1 this is $\mathrm{s} * \mathrm{s} = 1$ if $N \in 4\mathbb{N}$. But for $N \in 4\mathbb{N}_0 + 2$ this reads $\sigma(W) = \sigma(W_{1,2})$. Thus we first check the case $N = 2$. If $\sigma(W_{1,2}) = +1$ then ϱ_{s} is self-conjugate, otherwise it is not self-conjugate, i.e. $\mathrm{s} * \mathrm{s} = \mathrm{v}$. It is a result of Guido and Longo [14] that a conjugate morphism $\overline{\varrho}$ is given by

$$\overline{\varrho} = \mathrm{j} \circ \varrho \circ \mathrm{j}$$

where j is the antiautomorphism corresponding to the reflection $z \mapsto \overline{z}$ on the circle (PCT transformation). In our model, j is the extension of the antilinear Bogoliubov automorphism j_Θ,

$$\mathrm{j}_\Theta(B(f)) = B(\Theta f), \qquad \Theta f = \Theta\left((f^i)_{i=1,2}\right) = \left(\overline{f^i_{\mathrm{refl}}}\right)_{i=1,2},$$

where $f \in L^2(S^1; \mathbb{C}^2)$ and $\overline{f^i_{\mathrm{refl}}}(z) = \overline{f^i(\overline{z})}$ for $z \in S^1$. So we have a candidate $\overline{\varrho_{\mathrm{s}}} \equiv \overline{\varrho_V} = \varrho_{\Theta V \Theta}$. It is quite obvious that $\Theta P_{I_\pm} \Theta = P_{I_\mp}$ and that $\Theta f^i_p = f^i_p$, $p \in \frac{1}{2}\mathbb{Z}$, so it follows by antilinearity of Θ ($N = 2$)

$$\Theta V \Theta = -P_{I_-} + P_{I_+} + (-\mathrm{i}r^1 - \mathrm{i}R^1) - (t^2 - \mathrm{i}T^2).$$

It is not hard to see that this is

$$\Theta V \Theta = U_{1,2} V, \qquad U_{1,2} = 2|v^1_{\frac{1}{2}}\rangle\langle v^1_{\frac{1}{2}}| - \mathbf{1}, \qquad v^1_{\frac{1}{2}} = \frac{1}{\sqrt{2}}(f^1_{\frac{1}{2}} + f^1_{-\frac{1}{2}}).$$

Now $U_{1,2}$ is as in Corollary 3.6 so that we find $\mathrm{s} * \mathrm{v} * \mathrm{s} = \mathrm{s} * \mathrm{c} = 1$ for $N = 2$. Hence $\sigma(W_{1,2}) = -1$, so it follows $\mathrm{s} * \mathrm{c} = 1$ for all $N \in 4\mathbb{N}_0 + 2$. For the case $N \in 2\mathbb{N}_0 + 1$ the situation is different because ϱ_V then is not an automorphism. As discussed at the end of Section 5, the representation $\pi_{\mathrm{NS}} \circ \varrho_V$ (and, of course, also $\pi_{\mathrm{NS}} \circ \varrho_U \varrho_V$) decomposes, in restriction to $\mathcal{C}(\mathcal{K}, \Gamma)^+$, into two equivalent irreducibles corresponding to the spinor sector σ. So we find at first $\mathrm{v} * \sigma = \sigma$. Let us consider $\pi_{\mathrm{NS}} \circ \varrho_V^2$. We have $M_{V^2} = 2M_V = 2$, hence by Theorem 3.3 and Lemma 5.2 we conclude $\pi_{\mathrm{NS}} \circ \varrho_V^2 \simeq 2\pi_{\mathrm{NS}}$. In restriction to $\mathcal{C}(\mathcal{K}, \Gamma)^+$ this reads $\pi^+_{\mathrm{NS}} \circ \varrho_V^2 \oplus \pi^-_{\mathrm{NS}} \circ \varrho_V^2 \simeq 2(\pi^+_{\mathrm{NS}} \oplus \pi^-_{\mathrm{NS}})$. Our previous results admit only $\pi^+_{\mathrm{NS}} \circ \varrho_V^2 \simeq \pi^-_{\mathrm{NS}} \circ \varrho_V^2$ and hence we find $\sigma * \sigma = 1 + \mathrm{v}$. Summarizing we rediscover the WZW fusion rules.

Theorem 7.1 (Fusion Rules) *The basic (1), vector (*v*) and spinor (*s,c,σ*) sectors compose as follows. Dependent on the integer N, the fusion rules read*

$$\mathrm{v} * \mathrm{v} = \mathrm{s} * \mathrm{s} = \mathrm{c} * \mathrm{c} = 1, \qquad \mathrm{s} * \mathrm{c} = \mathrm{v}, \qquad N \in 4\mathbb{N}, \tag{37}$$

$$\mathrm{v} * \mathrm{v} = \mathrm{s} * \mathrm{c} = 1, \qquad \mathrm{s} * \mathrm{s} = \mathrm{c} * \mathrm{c} = \mathrm{v}, \qquad N \in 4\mathbb{N}_0 + 2, \tag{38}$$

$$\mathrm{v} * \mathrm{v} = 1, \qquad \sigma * \mathrm{v} = \sigma, \qquad \sigma * \sigma = 1 + \mathrm{v}, \qquad N \in 2\mathbb{N}_0 + 1. \tag{39}$$

We observe that if N is even all sectors are simple. For $N \in 4\mathbb{N}$ the fusion rules correspond to the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2$, for $N \in 4\mathbb{N}_0 + 2$ they correspond to $\mathbb{Z}_4$. If N is odd the spinor sector σ is not simple corresponding to the fact that ϱ_σ is not an automorphism; one obtains the Ising fusion rules.

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